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ASYMPTOTICS OF FALLING-DOWN SOLITARY WAVES

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At small Weber numbers and at Reynolds numbers close to the critical value, the shape of the free surface of a viscous layer falling down an inclined plane is known to be approximated by a Korteweg–de Vries-type equation with a solution in the form of a solitary wave [1–6].

In the present paper, an appropriate asymptotic solution of the complete set of Navier–Stokes equations and boundary conditions is constructed. An analysis of the leading expansion terms reveals the asymmetry of the solution, which imposes an additional solvability condition associated with group invariance of the problem. This condition is satisfied identically with respect to the group parameter by virtue of the dependence of the Reynolds number of the basic flow on the parameters of the corresponding solitary wave. The dependence shows that the branching is locally directed toward the subcritical region for a plane-parallel flow.

1. Formulation of the Problem in Mises-Type Variables. We consider a steady-state flow of a layer of an incompressible viscous liquid in a plane at an angle α with the horizontal. The coordinate system moves parallel to the bottom with wave velocity c. The coordinate origin is chosen at the unperturbed free boundary. The x and y axes are directed upstream and downstream, respectively. The flow bounded by a rigid bottom ($y = 1$) and a free surface [$y = h(x)$] obeys the Navier–Stokes equations. With allowance for the Nusselt ratio $\mathrm{Re}\,\mathrm{Fr}\sin\alpha = 3$ for a plane-parallel flow [2], these equations take the form

$$\mathrm{Re}\,(\psi_y\psi_{xy} - \psi_x\psi_{yy}) = -3 + \Delta\psi_y - \mathrm{Re}\,p_x; \tag{1.1}$$

$$\mathrm{Re}\,(-\psi_y\psi_{xx} + \psi_x\psi_{xy}) = 3\mathrm{cotan}\,\alpha - \Delta\psi_x - \mathrm{Re}\,p_y, \tag{1.2}$$

where $\psi(x,y)$ is the stream function, $\mathrm{Re} = Q/\nu$ is the Reynolds number, $\mathrm{Fr} = gH^3/Q^2$ is the Froude number, H is the unperturbed layer depth, and Q is the flow rate in the fixed coordinate system. At the free boundary and at the bottom, two dynamic and three kinematic conditions

$$\psi_{yy} - \psi_{xx} - \frac{4h'}{1-h'^2}\psi_{xy} = 0, \quad y = h(x); \tag{1.3}$$

$$-\mathrm{Re}\,p = 2\frac{1+h'^2}{1-h'^2}\psi_{xy} + \frac{\mathrm{We}}{3}\frac{h''}{(1+h'^2)^{3/2}}, \quad y = h(x); \tag{1.4}$$

$$\psi(x,h(x)) = 0, \qquad \psi(x,1) = c-1, \qquad \psi_y(x,1) = c \tag{1.5}$$

are satisfied. Here $\mathrm{We} = \mathrm{Re}\,\sigma H\,/\,Q^2$ is the Weber number and σ is the surface-tension coefficient.

We have to find a solitary-type solution of problem (1.1)–(1.5) that becomes a plane-parallel flow at infinity with the stream function

$$\Psi(\eta) = \frac{1}{2}\eta^3 + \frac{3}{2}\lambda\eta, \qquad \lambda = \frac{2}{3}c - 1, \tag{1.6}$$

where η is the Lagrangian coordinate which coincides with the Eulerian coordinate y at infinity. According to [1], we choose, in problem (1.1)–(1.5), the Lagrangian coordinate η related to the stream function by formula

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(1.6) as an independent transverse coordinate, and, as a dependent one, we choose the function $\omega(x,\eta)$ related to the Eulerian coordinate y by the formula

$$y(x,\eta) \equiv 1 + \int\limits_1^\eta \exp\left[-\left(\frac{\omega(x,t)}{\Psi'(t)}\right)_t\right] dt = \eta - \frac{\omega(x,\eta)}{\Psi'(\eta)} + O(\omega^2)$$

and subject to the conditions

$$\omega(x,1) = \omega_\eta(x,1) = 0. \tag{1.7}$$

In the case considered, the dimensionless wave velocity $c \approx 3$, and $\Psi'(\eta)$, therefore, is different from zero on the section $[0,1]$.

Conditions (1.5) now are satisfied identically. Omitting the pressure in the Navier–Stokes equations leads to the following integro-differential equation in the $0 < \eta < 1$ domain:

$$\begin{aligned}\omega_{\eta\eta\eta} &= 3\exp\left[-3\left(\frac{\omega}{\Psi'}\right)_\eta\right] - 3 + 9\left(\frac{\omega}{\Psi'}\right)_\eta - 2\Psi'\left(\frac{\omega}{\Psi'}\right)^2_{\eta\eta} \\ &+\left\{-3\operatorname{cotan}\alpha\, y_x + \left[\frac{2\Psi'}{y_\eta}\frac{1+y_x^2}{1-y_x^2}\left(\frac{y_x}{y_\eta}\right)_\eta - \frac{\mathrm{We}}{3}\frac{y_{xx}}{(1+y_x^2)^{3/2}}\right]_x\right\}\Big|_{\eta=0} y_\eta^3 \\ &+y_\eta^3\int\limits_0^\eta\left[(z_\eta y_x - z_x y_\eta)_x - \mathrm{Re}\,\Psi'^2\left(\frac{y_x}{y_\eta}\right)_{xx}\right]d\eta' - \mathrm{Re}\,\Psi'^2 y_{x\eta} \\ &+\mathrm{Re}\,\Psi'^2 y_\eta^2 y_x\left(\frac{y_x}{y_\eta}\right)_x - z_\eta y_\eta^2 y_x^2 - y_\eta^2\left[z - \frac{1}{y_\eta}\left(\frac{\Psi'}{y_\eta}\right)_\eta\right]_\eta + z_x y_\eta^3 y_x \\ &\left[z \equiv \Delta\psi = \frac{1}{y_\eta}\left(\frac{\Psi'}{y_\eta}\right)_\eta - \Psi'\left(\frac{y_x}{y_\eta}\right)_x + \frac{y_x}{y_\eta}\left(\frac{\Psi' y_x}{y_\eta}\right)_\eta\right].\end{aligned} \tag{1.8}$$

The problem is closed by the boundary condition (1.3) which takes the following form in new variables:

$$\omega_{\eta\eta} - \frac{2}{\lambda}\omega = -\frac{3}{2}\lambda y_\eta^2\left[\left(\frac{y_x}{y_\eta}\right)_x + \frac{3+y_x^2}{1-y_x^2}\frac{y_x}{y_\eta}\left(\frac{y_x}{y_\eta}\right)_\eta\right], \quad \eta = 0. \tag{1.9}$$

Problem (1.7)–(1.9) has the following advantage: under the assumption of smallness of the Re number and the cotangent of the slope angle, its linear part is a problem with constant coefficients. The latter was studied by the author in [7], where the necessary and sufficient conditions for its solvability were found, the Green function was constructed, and the corresponding estimates were given.

2. Construction of an Asymptotic Solution. In problem (1.7)–(1.9), we extend the longitudinal variable $\xi = \varepsilon x$ and represent its solution as a series

$$\omega(\xi,\eta) = \varepsilon^2\sum_{k=0}^{\infty}\omega_k(\xi,\eta)\varepsilon^k. \tag{2.1}$$

In addition, according to [1], we set

$$2/\lambda = 2 - \varepsilon^2, \tag{2.2}$$

under the assumption that the spectral parameter $2/\lambda$ is close to the eigenvalue of the linear problem. Equality (2.2) relates the wave velocity to the length scale ε as follows:

$$c = \frac{3}{2}(1+\lambda) = 3 + \frac{3\varepsilon^2}{4-2\varepsilon^2} = 3 + \frac{3}{4}\varepsilon^2 + O(\varepsilon^4).$$

The Reynolds number is regarded as a desired function of the problem parameters and we find it as a series $\mathrm{Re} = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots$. The Weber number is assumed to be a free parameter of the order of unity.

As a result, we obtain, from problem (1.7)–(1.9), the recurrent sequence of boundary-value problems of the form

$$\begin{gathered}\omega_{k,\eta\eta\eta} = f_k(\omega_0,\dots,\omega_{k-1}), \quad 0<\eta<1, \\ \omega_{k,\eta\eta} - 2\omega_k = \varphi_k(\omega_0,\dots,\omega_{k-1}), \quad \eta=0, \qquad \omega_k = \omega_{k,\eta} = 0, \quad \eta = 1,\end{gathered} \tag{2.3}$$

where f_k and φ_k are the known nonlinear operators. Satisfaction of the condition

$$\varphi_k(\omega_0,\dots,\omega_{k-1}) + \int\limits_0^1 f_k(\omega_0,\dots,\omega_{k-1})(1-\eta^2)\,d\eta = 0 \tag{2.4}$$

is the necessary and sufficient condition for solvability of problem (2.3). When it is satisfied, the general solution of problem (2.3) can be written as

$$\omega_k(\xi,\eta) = \Omega_k(\xi,\eta) + C_k(\xi)v_0(\eta), \tag{2.5}$$

where $v_0(\eta) = (3/2)(1-\eta)^2$ is the solution of the homogeneous problem (2.3); $C_k(\xi)$ is an arbitrary function and $\Omega_k(\xi,\eta)$ is the particular solution of problem (2.3) subject to the condition $\Omega_k(\xi,0)=0$.

Ter-Krikorov [1] found the following formula for the first-order terms of expansion (2.1):

$$\omega_0(\xi,\eta) = C_0(\xi)v_0(\eta), \qquad \omega_1(\xi,\eta) = C_0'(\xi)v_1(\eta) + C_1(\xi)v_0(\eta)$$
$$\Big[C_0(\xi) = \frac{3}{8}\cosh^{-2}\frac{\xi}{4}, \qquad v_1(\eta) = \frac{\operatorname{cotan}\alpha}{16}(\eta^5 - 2\eta^3 + \eta)\Big].$$

The solvability condition (2.4) of a nonhomogeneous problem for definition of $\omega_1(\xi,\eta)$ is satisfied by choosing $R_0 = (5/6)\operatorname{cotan}\alpha$. This corresponds to a critical value of Re at which the plane-parallel flow becomes unstable [2, 3]. The function $C_0(\xi)$ is found, up to the translation, from the solvability condition (2.4) of the problem for determination of $\omega_2(\xi,\eta)$, which is of the form of the Korteweg–de Vries stationary equation

$$4C_0'' + 4C_0^2 - C_0 = 0, \tag{2.6}$$

if one sets $R_1 = 0$.

In finding the subsequent terms of the series (2.1), the arbitrary functions $C_k(\xi)$ $(k=1,2,\dots)$ entering (2.5) are determined from the solvability condition (2.4) of the boundary-value problem for the $(k+2)$th term of the expansion being the $C_0(\xi)$-linearized equation (2.6):

$$4C_k'' + (8C_0 - 1)C_k = F_k(C_0,\dots,C_{k-1}), \qquad k=1,2,\dots\,. \tag{2.7}$$

With allowance for the equation for $C_0(\xi),\dots,C_{k-1}(\xi)$ and their differential consequences, the operator $F_k(C_0,\dots,C_{k-1})$ is represented as a polynomial in its own arguments and their first derivatives. Thus, under the condition of an exponential decrease in the functions $C_0(\xi),\dots,C_{k-1}(\xi)$, the right-hand side of Eq. (2.7) decreases exponentially as well. In this case, the orthogonality condition

$$\int\limits_{-\infty}^{+\infty} F_k(C_0,\dots,C_{k-1})C_0'(\xi)\,d\xi = 0 \tag{2.8}$$

is necessary and sufficient for the existence of an exponentially decreasing solution of Eq. (2.7). The solution of Eq. (2.7) is written in the form

$$\begin{gathered}C_k(\xi) = KF_k(C_0,\dots,C_{k-1}) + \nu_k C_0'(\xi), \\ KF \equiv V_1(\xi)\int\limits_0^{\xi} V_0(\xi')F(\xi')\,d\xi' + V_0(\xi)\int\limits_{\xi}^{+\infty} V_1(\xi')F(\xi')\,d\xi',\end{gathered} \tag{2.9}$$

where $V_0(\xi) = -V_1(\xi)\int V_1(\xi')^{-2}\,d\xi'$ and $V_1(\xi) = -(64/3)C_0'(\xi)$ are the even and odd solutions of the homogeneous equation (2.7). Note that the operator K conserves the evenness or oddness with respect to ξ.

The arbitrary constants ν_k in (2.9) describe the translation of the coordinate system along the ξ axis by the quantity $\varepsilon^k \nu_k$.

An important circumstance in constructing the series of perturbations is the possibility of satisfying conditions (2.8) for $k = 1, 2, \dots$. In the case of symmetric solitons, for example, solitary waves in an ideal fluid [8], the orthogonality conditions turn out to be satisfied on a space of even functions automatically. For asymmetric solutions of variational problems, conditions (2.8) are fulfilled identically with respect to all parameters as a consequence of certain conservation laws [9]. In the general nonconservative case, these conditions establish an additional link between the governing parameters of the problem and the solution parameters, i.e., in the problem considered, condition (2.8) is satisfied at each kth step owing to the determination of the coefficient R_{k+1} of the Re expansion.

As an example, we find the subsequent term $\omega_1(\xi, \eta)$ of expansion (2.1). The desired function $C_1(\xi)$ in the representation (2.5) is defined from the solvability condition (2.4) of a boundary-value problem for determination of $\omega_3(\xi, \eta)$, the right-hand parts of which contain the terms $\omega_0(\xi, \eta)$, $\omega_1(\xi, \eta)$, and $\omega_2(\xi, \eta)$. The latter term is of the form of (2.5), where

$$\Omega_2(\xi, \eta) = C_0''(\xi) v_{21}(\eta) + C_0(\xi)^2 v_{22}(\eta) + C_1'(\xi) v_1(\eta),$$

$$v_{21}(\eta) = \frac{5\cot\mathrm{an}^2\alpha}{10752}\eta^9 + \frac{\cot\mathrm{an}^2\alpha}{896}\eta^7 - \frac{7\cot\mathrm{an}^2\alpha}{768}\eta^5 - \frac{1}{4}\eta^4 + \left(\frac{3}{2} + \frac{5\cot\mathrm{an}^2\alpha}{384}\right)\eta^3 - \frac{9}{4}\eta^2 + \left(1 - \frac{59\cot\mathrm{an}^2\alpha}{10752}\right)\eta,$$

$$v_{22}(\eta) = \frac{3}{4}(\eta^2 + 1)\arctan\eta + \frac{1}{(\eta^2+1)^2}\left[-3\eta^6 + \frac{34 - 3\pi}{8}\eta^5 - 6\eta^4 + \frac{32 - 3\pi}{4}\eta^3 - 3\eta^2 - \frac{2 + 3\pi}{8}\eta\right].$$

We write the right-hand side of Eq. (2.8) for $C_1(\xi)$ as

$$F_1(\xi) = \left(a_{11}(\alpha, \mathrm{We}) + \frac{8}{5}R_2\right) C_0'(\xi) + a_{12}(\alpha, \mathrm{We})\, C_0(\xi)\, C_0'(\xi).$$

The orthogonality condition (2.8) for $F_1(\xi)$ is satisfied if we set

$$R_2(\alpha, \mathrm{We}) = -\frac{535}{25{,}088}\cot\mathrm{an}\,\alpha - \frac{25}{6{,}054{,}048}\cot\mathrm{an}^3\alpha - \frac{25}{1512}\mathrm{We}. \tag{2.10}$$

The function $C_1(\xi)$ is then found according to formula (2.9). As $F_1(\xi)$ is the odd function, the function $C_1(\xi)$ will be odd as well.

3. Invariant Property of Orthogonality Conditions. In finding each subsequent expansion term in formula (2.9), an additional free parameter ν_k appears, which is responsible, as noted above, for translation of the coordinate system. Now it is important to understand how the parameters $\nu_1, \dots, \nu_{n-2}$ enter the orthogonality conditions (2.8) for $F_{n-1}(C_0, C_1, \dots, C_{n-2})$ and how they affect the determination of the coefficient R_n.

Theorem. *The coefficients R_n found from the orthogonality conditions (2.8) for $F_{n-1}(C_0, C_1, \dots, C_{n-2})$ are not dependent on the parameter $\nu_1, \dots, \nu_{n-2}$.*

Proof. This is performed by induction with respect to n. Let us select the dependence of the functions C_k on the parameters ν_j. We obtain by successive calculations that

$$C_0(\xi) \equiv U_0(\xi), \qquad C_1(\xi; \nu_1) = U_1(\xi) + \nu_1 U_0'(\xi),$$
$$C_2(\xi; \nu_1, \nu_2) = U_2(\xi) + \nu_1 U_1'(\xi) + \frac{\nu_1^2}{2} U_0''(\xi) + \nu_2 U_0'(\xi). \tag{3.1}$$

Here $U_k(\xi) \equiv K F_k(U_0, \dots, U_{k-1})$ are functions that are not dependent on the parameters ν_j. To write the general term, it is convenient to consider $C_k(\xi; \nu_1, \dots, \nu_k)$ as generating functions in variables $\nu_1, \dots, \nu_k$ (see [10]). We introduce the operators Φ_i of the variables ν_j by the formula

$$\Phi_i\left(\prod_{j=1}^{s} \nu_j^{l_j}\right) = \nu_i \prod_{j=1}^{s} \nu_j^{l_j} \Big/ \left(1 + \sum_{j=1}^{s} l_j\right).$$

For $C_k(\xi;\nu_1,\ldots,\nu_k)$, the following recurrent expression, which is also determined by induction,

$$C_k(\xi;\nu_1,\ldots,\nu_k)=U_k(\xi)+\sum_{l=1}^{k-1}\Phi_l C'_{k-l,\xi}(\xi;\nu_1,\ldots,\nu_{k-l})+\nu_k C'_0(\xi) \tag{3.2}$$

is then true.

Formulas (3.1) are the basis for induction. We assume that, for $k=0,\ldots,n$, the functions $C_k(\xi)$, which are defined as the solutions of Eqs. (2.7), have the representations (3.2). If one substitutes these representations into the general formula for operator $F_{n+1}(C_0,\ldots,C_n)$, after some manipulations, with allowance for Eqs. (2.7) for $C_k(\xi;\nu_1,\ldots,\nu_k)$ $(k=1,\ldots,n)$ and their differential consequences, we obtain

$$F_{n+1}(C_0,\ldots,C_n)=F_{n+1}(U_0,\ldots,U_n)+L\sum_{l=0}^{n}\Phi_l C'_{n+1-l}, \tag{3.3}$$

where L is the linear operator on the left-hand side of Eq. (2.7). Since the operator L is self-conjugate and $C'_0(\xi)$ is the element of its kernel, it follows from (3.3) that

$$\int_{-\infty}^{+\infty} F_{n+1}(C_0,\ldots,C_n)C'_0(\xi)\,d\xi=\int_{-\infty}^{+\infty} F_{n+1}(U_0,\ldots,U_n)C'_0(\xi)\,d\xi.$$

The integral on the right-hand side is not dependent on the parameters ν_j. If one substitutes the relation (3.3) into formula (2.9), and takes into account that operators K and L are mutually inverse, one obtains that the solution $C_{n+1}(\xi;\nu_1,\ldots,\nu_{n+1})$ of Eq. (2.7) also has the structure (3.2). The theorem is proved.

It is convenient to choose the coordinate system with the requirement for passing the y axis through a maximum elevation of the free boundary. To this requirement corresponds the condition $\omega_x(0,0)=0$, which is equivalent to the conditions $C'_k(0)=0$ satisfied for $\nu_k=0$ $(k=1,2,\ldots)$. One can prove by induction that, in this case, each term $\omega_k(\xi,\eta)$ has a definite evenness in ξ, which coincides with the evenness of its ordinal number k. It has been revealed that the coefficients R_k, which were found from the orthogonality conditions (2.8), are equal to zero for odd k. Thus, the Re expansion occurs only in even powers of ε.

4. Conclusions. In original variables, the asymptotic solution obtained takes the form

$$\psi(x,y)=\Psi(\eta(x,y))=\Psi(y)+\varepsilon^2\omega_0(\varepsilon x,y)+\varepsilon^3\omega_1(\varepsilon x,y)+O(\varepsilon^4),$$

$$h(x)=y(x,\eta)\Big|_{\eta=0}=-\varepsilon^2C_0(\varepsilon x)-\varepsilon^3C_1(\varepsilon x)+O(\varepsilon^4).$$

The zero expansion coefficients are found in full accordance with the results of the approximate theory of film flows [2–6] for similar solutions of the governing parameters.

The relations $\mathrm{Re}=(5/6)\mathrm{cotan}\alpha+\varepsilon^2R_2(\alpha,\mathrm{We})+O(\varepsilon^4)$, where the coefficient $R_2(\alpha,\mathrm{We})$ is negative at all values of the slope angle and Weber numbers [see formula (2.10)], shows that, for small ε, the branch of the solutions is directed toward the subcritical domain for plane-parallel flow. This means that, for the constructed family, a "continuous" transition to a wave flow regime in the form of a gradual loss of stability of the basic flow is not possible in principle.

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